**Special Relativity Kinematics**

**Particle Kinematics**

Now we’d like to introduce space-time position, velocity, momentum, etc., vectors, in preparation for writing down the dynamical equations in an invariant way. So first, the position vector.



This is given by:



It is invariant w/r to Lorentz transformation since:



And therefore its magnitude is invariant w/r to Lorentz transformations – as we know actually since this is:



which we previously observed was relativistically invariant. Now let’s define the space-time displacement vector:



And let’s calculate the magnitude of this vector, which is invariant as it must be:



If we specialize to the rest frame of the particle then we’ll have:



So we have the general result,



Before moving on, let’s note that for photons, Δs2 = 0 always, which implies that dτ = 0 so that there is no ‘proper’ time for a photon. Therefore this result only applies to particles moving at less than the speed of light, i.e. to time-like vectors. Now let’s define the space-time velocity. It is the rate of change of a particle’s space-time position w/r to time in its own rest frame time.



And this is given by:



The spatial part of the vector will reduce to the usual velocity vector at low speeds. Since this is a vector, it is invariant w/r to Lorentz transformations, and in particular its magnitude is invariant – as we discussed previously is true for any vector. Let’s work out its magnitude…we have (remember dot product of two time-like basis vectors is -1):



and so,



So we can see that the norm of the 4-velocity is –c2. Thus, all objects seem to travel through space-time at the same ‘rate’, which is the speed of light. This fact provides another point of view on the time-dilation. The rate at which one travels through space-time is constant, and thus as you speed up (increasing your rate of travel through space), your rate of travel through time must slow down (this is time-dilation) to keep your rate of travel through space-time constant. Finally here let’s observe that the space-time velocity of a photon would be ∞ since its rest time dτ = 0. Next let’s consider the space-time momentum. We can define the space-time momentum vector as:



This is given by:



(where we’ve peaked ahead and recognized the time-like component of is energy/c, and the spatial component is just the spatial momentum) For the magnitude (squared) of the vector we have:



So we have:



Here’s a useful formula. Say a particle is moving with speed **v** and an observer with speed **u**. What energy does the observer measure for the particle? We can take advantage of an invariant. Consider the scalar product: ∙. What is this equal to? The claim is that this is equal to the energy of the particle as measured by someone in the observer’s frame of reference. Let’s see. We can do a Lorentz transformation of into the frame moving with speed **u** (β = u/c):



The energy in this boosted frame is the zeroth component term:



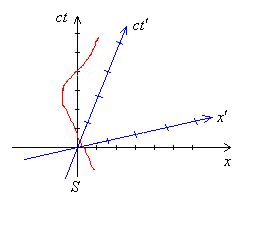
So indeed we see:



This reduces to what we expect in special cases. When **u** = 0, we get E, and when **u** = **v** = **p**/m, we get mc2.

**Example: velocity transformation laws**

The geometrical properties of these space-time vectors makes determining transformation laws easier in some respects. For instance, let’s determine the velocity addition law. So suppose we have an object thrown with speed **v** in reference frame S. What is its speed **v**′ in reference frame S′ moving with speed **u** (in the x-direction) w/r to S?



Well, since  is a space-time vector, its components will transform according to:



Anyway, so writing these components out we get:



which implies,



**Example**

A spacecraft is moving at a velocity of **v** = 0.6c relative to a space station. Inside the spacecraft, Particle A moves at **v**A = 0.5c+ 0.7c, and Particle B moves at **v**B = 0.7c+ 0.5c​, both velocities measured relative to the spacecraft. Which particle has the larger velocity in the space station's frame?

So we can use the velocity transformation laws. Let **v**´ be the velocity of the particles w/r to the station, which moves with speed **u** = -0.6c w/r to the space ship.



So the velocity magnitudes are:



So particle B has the larger velocity.

**Photon Kinematics**

We can write a space-time displacement vectors for the photon. The 4-velocity vector is undefined for a photon however, because its proper time does not progress. Instead we just differentiate the displacement w/r to something else, like arc length perhaps (an affine parameter in general). The 4-momentum can be defined for a photon, however, using the equation E = √[(mc2)2 + (pc)2] = √[(0) + (pc)2]. Below we use p to denote the magnitude of the spatial momentum |**p**|.



The magnitudes of these vectors would be:



So yeah.